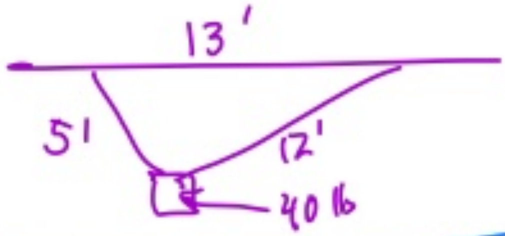
Warm-ups

① Find a vector of length 5 that is in the same direction as $(2,1,-1,3)$

② A 40lb weight is suspended from a horizontal beam by two wires attached to the beam at 2 points that are 13 ft apart. One wire is 5ft long, and the other is 12 ft long. How much tension (force) is in each wire?



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① Length of $(2,1,-1,3) = |(2,1,-1,3)| = \|(2,1,-1,3)\|$ ↙ also called "magnitude" or "norm" of the vector

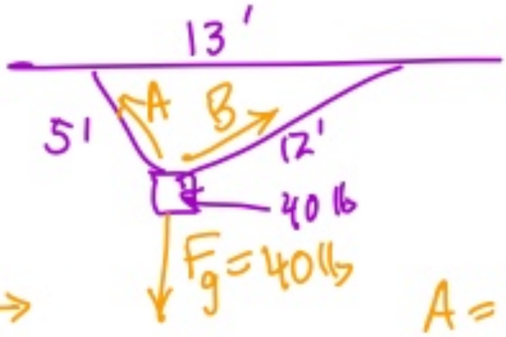
$= \sqrt{4+1+1+9} = \sqrt{15}$.

$\Rightarrow$ length of $\boxed{\frac{5}{\sqrt{15}}(2,1,-1,3)}$ is $\left\|\frac{5}{\sqrt{15}}(2,1,-1,3)\right\|$

$= \left|\frac{5}{\sqrt{15}}\right| \|(2,1,-1,3)\| = \frac{5}{\sqrt{15}} \cdot \sqrt{15} = 5.$

Property of |vector|: If $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then $|cv| = |c||v|$.

②



What must be true:

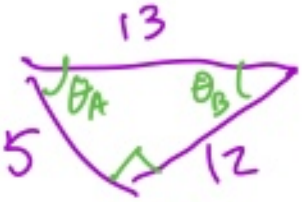
$A + B + F_g = 0$ Balance equation

$A = (A_1, A_2)$, $B = (B_1, B_2)$

$F_g = (0, -40)$

Balance equation: $A + B + F_g = 0$

$$\star \begin{cases} A_1 + B_1 = 0 \\ A_2 + B_2 - 40 = 0 \end{cases}$$

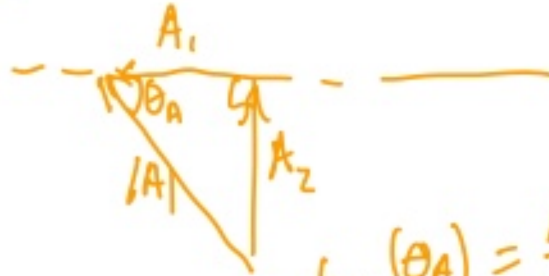


$\tan\theta_A = \frac{12}{5}$

$\tan\theta_B = \frac{5}{12}$

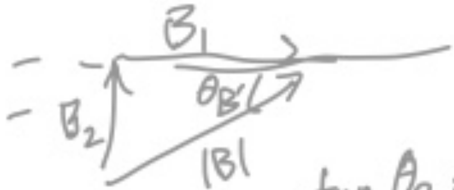
$5^2 + 12^2 = 13^2$

$25 + 144 = 169$



$\tan(\theta_A) = \frac{A_2}{A_1} = \frac{-12}{5}$

negative.



$\tan\theta_B = \frac{B_2}{B_1} = \frac{5}{12}$

Back to $\star$ $A_2 = \frac{-12}{5}A_1$, $B_2 = \frac{5}{12}B_1$

$A_1 + B_1 = 0 \longrightarrow B_1 = -A_1$

$\frac{-12}{5}A_1 + \frac{5}{12}B_1 - 40 = 0$

$\Downarrow$

$\frac{12}{5} + \frac{5}{12} = \frac{144+25}{60} = \frac{169}{60}$

$\frac{-12}{5}A_1 - \frac{5}{12}A_1 = 40$

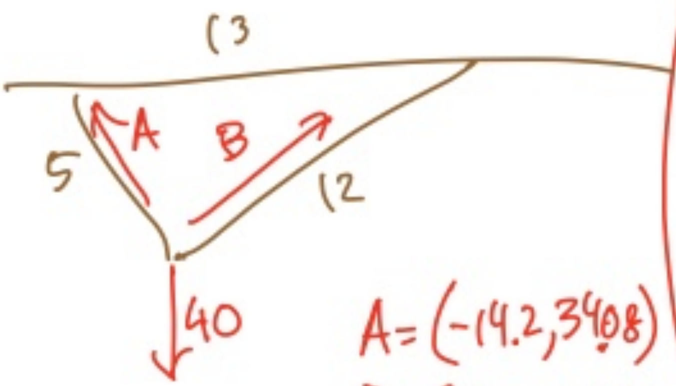
$\Rightarrow -\frac{169}{60}A_1 = 40$

$\Rightarrow A_1 = \frac{-40 \cdot 60}{169} = -\frac{2400}{169} = -14.2$

$B_1 = \frac{2400}{169} = 14.2$

$A_2 = -\frac{12}{5}A_1 = -\frac{12}{5}\left(\frac{-2400}{169}\right) = 34.08$

$B_2 = \frac{5}{12}B_1 = \frac{5}{12}(14.2) = 5.92$



$A = (-14.2, 34.08)$

$B = (14.2, 5.92)$

Magnitude of A = tension in A wire

$= \sqrt{(14.2)^2 + (34.08)^2} =$ ____

magn. of B = tension in B wire $= \sqrt{(14.2)^2 + (5.92)^2} =$ ____

The dot product of vectors

If $v, w \in \mathbb{R}^n$, $v \cdot w \in \mathbb{R}$ (a scalar),

defined by $v \cdot w = (v_1, v_2, \dots, v_n) \cdot (w_1, w_2, \dots, w_n)$

$$= v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

example: $(-1, 2, 7) \cdot (4, 1, 0) = (-1)4 + 2(1) + 7(0)$

$$= -4 + 2 = \boxed{-2}.$$

Occasionally, you will see $\langle v, w \rangle = v \cdot w$.

Properties of Dot product: $\forall\, v, w, z \in \mathbb{R}^n$, $c \in \mathbb{R}$

1. $v \cdot v = (v_1, v_2, \dots, v_n) \cdot (v_1, v_2, \dots, v_n)$ positivity

   $= v_1^2 + v_2^2 + \dots + v_n^2 = |v|^2 \geq 0$

   ($= 0$ only in the case where

   $v = (0, 0, \dots, 0)$ ← we will often denote the zero vector by 0.

   (We can tell from context if we mean $0 \in \mathbb{R}$ or $0 = (0, 0, \dots, 0)$.)

2. $v \cdot w = w \cdot v$ Commutative property of $\cdot$

   (symmetry property of $\cdot$)

3. $\underbrace{(\underbrace{cv}_{\text{vector}}) \cdot \underbrace{w}_{\text{vector}}}_{\text{scalar}} = \underbrace{c}_{\text{scallar}} (\underbrace{v \cdot w}_{\text{scalar}})$ homogeneity

   (similar to assoc. property)

Similarly $v \cdot (cw) = c(v \cdot w)$

Also works with subtraction

4) $(v+w)\cdot z = v\cdot z + w\cdot z$ linearity (distributive property)

similarly

$v\cdot(w+z) = v\cdot w + v\cdot z$

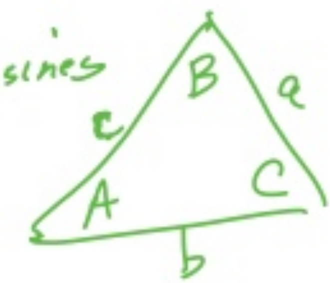
Geometric Formula for Dot Product.

$$v\cdot w = |v|\,|w|\cos\theta$$

Measure θ in the plane containing the two vectors.

Why is this true?

Geometry formula
Law of Cosines



$$a^2 = b^2 + c^2 - 2bc(\cos A)$$

$$|w-v|^2 = |w|^2 + |v|^2 - 2|w|\,|v|\cos\theta$$

$$\Rightarrow (w-v)\cdot(w-v) = w\cdot w + v\cdot v - 2|w|\,|v|\cos\theta$$

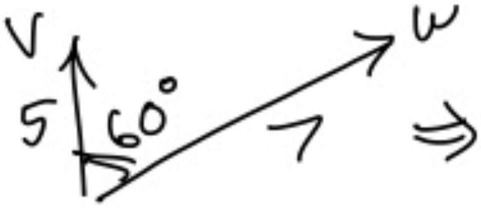
$$(w-v)\cdot w - (w-v)\cdot v = w\cdot w + v\cdot v - 2|w|\,|v|\cos\theta$$

$$w\cdot w - v\cdot w - \{w\cdot v - v\cdot v\} = w\cdot w + v\cdot v - 2|w|\,|v|\cos\theta$$

$$\cancel{w\cdot w} - v\cdot w - \underbrace{w\cdot v}_{v\cdot w} + \cancel{v\cdot v} = \cancel{w\cdot w} + \cancel{v\cdot v} - 2|w|\,|v|\cos\theta$$

$$-2v\cdot w = -2|w|\,|v|\cos\theta$$

$$\boxed{v\cdot w = |v|\,|w|\cos\theta}$$



$$\Rightarrow \quad v\cdot w = 5\cdot 7\cdot \cos(60°)$$
$$= 5\cdot 7\cdot \tfrac{1}{2} = \boxed{\tfrac{35}{2}}.$$

Note $\cos\theta = 0$ if $\theta = 90°$ or $-90°$.

$\Rightarrow$ **$v\cdot w = 0$ if and only if the vectors v & w are perpendicular.**

eg: $(5,-1,7,2)$ and $(1,5,-2,7)$ are perpendicular.

Why?: $(5,-1,7,2)\cdot(1,5,-2,7) = 5(1)+(-1)(5)+7(-2)+2\cdot 7$
$= 0.$

Ex) Find the angle between the vectors $\overset{v}{(1,3,-1,2)}$ and $\underset{w}{(0,1,1,1)}$.

$$v\cdot w = |v|\,|w|\cos\theta$$

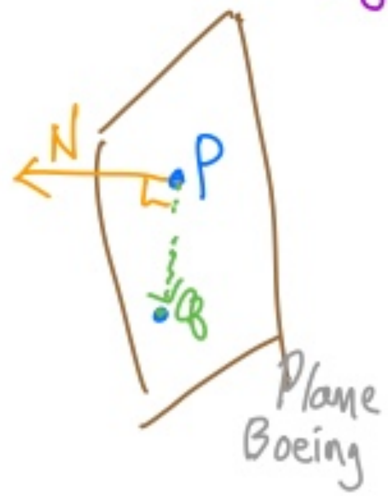
$$\Rightarrow \quad 1\cdot 0 + 3\cdot 1 + (-1)\cdot 1 + 2\cdot 1 = 4 = \left(\sqrt{1+9+1+4}\right)\left(\sqrt{1+1+1}\right)\cos\theta$$

$$\Rightarrow 4 = \sqrt{15}\sqrt{3}\cos\theta$$
$$\left(\begin{matrix}\sqrt{3\cdot 5}\\ = 3\sqrt{5}\end{matrix}\right)$$

$$\Rightarrow \cos\theta = \frac{4}{3\sqrt{5}} \Rightarrow \boxed{\theta = \arccos\left(\frac{4}{3\sqrt{5}}\right)}$$

Using the dot product in equations of planes:
(hyperplanes)

perpendicular
ortogonal



N = a normal vector to the plane.

P = a point on the plane.

Equation for q = ?
(any point on plane)

$$\vec{N} \cdot \overrightarrow{pq} = 0$$

Example: Find the equation of the plane in $\mathbb{R}^3$ that contains the point $(1,-1,0)$ and is perpendicular to the vector $(2,1,-8)$.

$$p = (1,-1,0) \quad \vec{N} = (2,1,-8)$$

$$q = (x,y,z)$$

$$\vec{N} \cdot \overrightarrow{pq} = 0$$

$$(2,1,-8) \cdot (q - p) = 0$$

$$(2,1,-8) \cdot ((x,y,z) - (1,-1,0)) = 0$$

$$(2,1,-8) \cdot (x-1, y+1, z) = 0$$

$$\Rightarrow 2(x-1) + 1(y+1) + (-8)(z) = 0$$

$$\Rightarrow 2x-2+y+1-8z=0$$

$$\boxed{2x+y-8z=1}$$

In general Plane equation:

$$Ax+By+Cz=D$$

Note (A,B,C) is a vector orthogonal to the plane.

(other dimensions) $Ax+By+Cz+Dw=E$ in $\mathbb{R}^4$

$(A,B,C,D) \perp$ hyperplane.

Another form:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

(x_0,y_0,z_0) point on plane

(A,B,C) vector normal to plane.

$$A(x-x_0)+B(y-y_0)=0$$